Diffusion of inhomogeneous vortex tangle and decay of superfluid turbulence

Sergey K. Nemirovskii
Institute of Thermophysics, Lavrentyev Avenue 1, 630090 Novosibirsk, Russia
and Novosibirsk State University, Pirogova 2, 630090 Novosibirsk, Russia

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The theory of inhomogeneous superfluid turbulence is developed on the basis of kinetics of merging and splitting vortex loops. Vortex loops composing the vortex tangle can move as a whole with some drift velocity depending on their structure and length. The flux of length, energy, momentum, etc., executed by the moving vortex loops takes place. The situation here is exactly the same as in usual classical kinetic theory, with the difference being that the "carriers" of various physical quantities are not the point particles but extended objects (vortex loops), which possess an infinite number of degrees of freedom, with highly involved dynamics. We suggest to complete our investigation, based on the supposition that vortex loops have a Brownian structure, with the only degree of freedom being, lengths of loops l. This concept allows us to study the dynamics of the vortex tangle on the basis of the kinetic equation for the distribution function $n(l,t)$—the density of a loop in the space of their lengths. Imposing the coordinate dependence on the distribution function $n(l,\mathbf{r},t)$ and modifying the "kinetic" equation with regard to an inhomogeneous situation, we are able to investigate various problems on the transport processes in superfluid turbulence. In this paper, we evaluate the flux of the vortex line density $\mathcal{L}(x,t)$ due to the gradient of this quantity. The corresponding evolution of quantity $\mathcal{L}(x,t)$ obeys the diffusion type equation, as it can be expected from dimensional analysis. The diffusion coefficient is arrived at from calculation of the (size-dependent) free path and drift velocity of the vortex loops, and takes the value 2.2κ, which exceeds approximately 20-fold the value obtained in early numerical simulation. We discuss the probable reason for this large discrepancy. We use the diffusion equation to describe the decay of the vortex tangle at a very low temperature. Comparison with recent experiments on decay of the superfluid turbulence is presented.

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I. INTRODUCTION AND SCIENTIFIC BACKGROUND

The idea that an inhomogeneous vortex tangle evolves in a diffusivelike manner appeared quite long ago. Thus, the authors of paper,[1] who observed the regions of high vortex line densities $\mathcal{L}(\mathbf{r},t)$—"plugs" in the channel with the counterflowing He II proposed that this phenomenon appeared due to diffusion of quantity $\mathcal{L}(\mathbf{r},t)$. An attempt to theoretically describe these processes was made in Refs. 1 and 2, where it was proposed to introduce the term proportional $\nabla^2\mathcal{L}(\mathbf{r},t)$ into the Vinen equation. The authors were not able to restore the value of the diffusion coefficient from experimental data (these results are reviewed and discussed in Ref. 3). Both the diffusion process and the diffusion coefficient we obtained from the so-called nonequilibrium thermodynamics principles developed earlier by the authors in series of works (see Ref. 4 and references therein). In the paper[5] the spatial diffusion of an inhomogeneous vortex tangle had been studied numerically (see Fig. 1). Analyzing their results, the authors determined the diffusion constant to be on the order of 0.1κ (κ is the quantum of circulation). Dynamics of the inhomogeneous vortex tangle had been studied numerically also in the paper.[6] However, since the authors studied the dilute tangle, they observed the "ballistic" regime, rather than pure diffusion. Escape of the vortex rings in the ballistic regime had been observed experimentally in work.[7]

In addition to the self-interest, the diffusion problem is related to the hypothetical close connection between classical (hydrodynamic) and superfluid turbulence. Recently, the idea that chaotic set of quantum vortices can mimic classical turbulence, or at least reproduce many main features, is actively being developed.[8–13] In principle, this idea had been discussed early (see, e.g., famous textbook by Frisch[14]), as an alternative version of the problem of classical turbulence. But only now, when the new powerful experimental methods

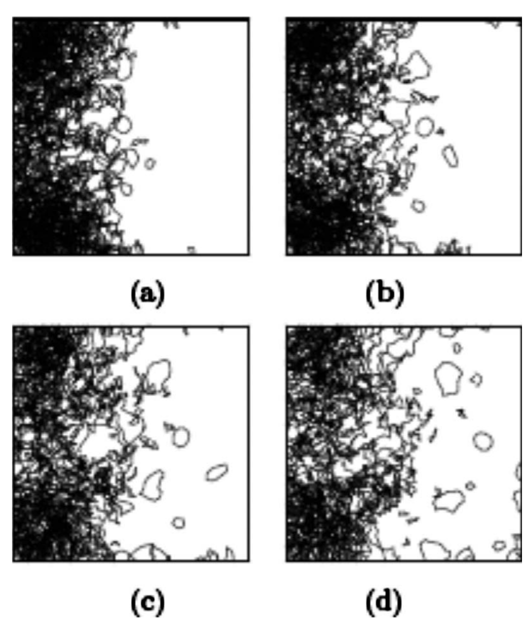

FIG. 1. Diffusion of a vortex tangle at (a) $t=0$ s, (b) $t=10.0$ s, (c) $t=20.0$ s, and (d) $t=30.0$ s [Tsubota *et al.* (Ref. 5)]. It is clearly seen as the near border loops can leave the volume carrying the line length, energy, momentum, etc.

in quantum fluids exist, this idea can be checked experimentally and it appears to be very attractive.

One of the starting points, which resulted in this activity, was the fact that vortex tangle decays at zero temperature, when the apparent mechanism of dissipation (mutual friction) is absent. Numerical and experimental observations of decay of the tangle at zero temperature are presented in the papers.[15–18] The physical mechanisms of the dissipation can be various, some approaches and ideas such as a cascadelike breakdown of the loops, Kelvin waves cascade, acoustic radiation, reconnection loss, etc., have been discussed in detail in recent review.[19] It is remarkable that all of these mechanisms are realized only on a very small scale. Therefore, it is natural to suggest that the Kolmogorov cascade occurs with the flow of energy, just as in the classical turbulence. The mechanism of the vortex tangle spreading, with the subsequent degeneration, is usually ignored. In fact, the contribution of diffusion had been discussed in the cited experimental works but based on the value 0.1κ for the diffusion constant obtained in Ref. 5, the authors concluded that this small diffusion coefficient did not lead to correct time of decay. Appreciating the significance of the challenging problem of classical turbulence, it should be expressed that the idea to describe the classical (hydrodynamic) turbulence with the use of a set of quantum vortices is indeed very essential and deserves scrupulous study. It is particular important to come back to the question about diffusivelike attenuation of the superfluid turbulence.

In this present paper, we develop the theory describing the evolution of an inhomogeneous vortex tangle on the basis of kinetics of the merging and breaking down vortex loops. We show that the evolution of a weakly inhomogeneous vortex tangle obeys the diffusion equation with the coefficient equal to about 2.2κ, which exceeds approximately 20 times the value obtained in Ref. 5. We present arguments that the diffusion constant would be significantly underestimated in Ref. 5, due to especial procedure used by the authors. We utilize the diffusion equation to estimate the contribution of diffusionlike attenuation of the vortex tangle at a very low temperature. Comparison is made with the recent experiments on decay of the superfluid turbulence.[16–18]

II. EVOLUTION OF INHOMOGENEOUS VORTEX TANGLE

A. Gaussian model

Vortex loops composing the vortex tangle can move as a whole with some drift velocity V_l depending on their structure and their length. The flux of the line length, energy, momentum, etc., executed by the moving vortex loops takes place. In the case of inhomogeneous vortex tangle the net flux due to the gradient of concentration of the vortex line density appears. The situation here is exactly the same as in classical kinetic theory with the difference being that the "carriers" are not the point particles but the extended objects (vortex loops), which possess an infinite number of degrees of freedom with very involved dynamics. In addition, while collision (or self-intersection) of elements of filaments the so-called reconnection of the lines occurs, loops either merge or split, losing their individuality and turning into other loops. The full statement of this problem requires some analog of the secondary quantization method for extended objects, or the string field theory, the problem of incredible complexity. Clearly, this problem can be hardly resolved in the near future. Some approach crucially reducing the number of degrees of freedom is required.

We offer to fulfill investigation basing on the supposition that vortex loops have the Brownian or random walking structure (see Refs. 20–22). In fact, the method to work with one-dimensional singularities (from polymer chains to cosmic strings[23–25]) as with random walking objects is widely accepted. This can be motivated by the following consideration. Because of the huge number of random collisions (Ref. 21) the structure of any loop is determined by numerous reconnections. Therefore, any loop consists of small uncorrelated parts, which "remember" previous collisions. This fact allows us to consider the loop as a random walk. The main mathematical tool to describe the random-walk structure is the Wiener distribution (see, e.g., Refs. 23 and 24) for the probability $\mathcal{P}(\{\mathbf{s}(\xi,t)\})$ to find some particular configuration $\{\mathbf{s}(\xi,t)\}$. From now on we will use the following notation. The elements of a vortex line are described with the use of a function $\mathbf{s}(\xi,t)$, which is the time-dependent radius vector of the points resting on the loop. Variable ξ labels the points of the loop. For the sake of convenience, we chose to make variable ξ equal to the arc length, $(0 \le \xi \le l)$. The first and second derivatives of $\mathbf{s}(\xi,t)$ with respect to variable ξ are just the tangent vector $\mathbf{s}'(\xi_1)$ and vector of curvature $\mathbf{s}''(\xi_1)$. The pure Wiener distribution has some deficiencies for describing real vortex filament. The most apparent one is that Wiener distribution does not have a finite average $\langle \mathbf{s}'(\xi,t)\mathbf{s}'(\xi,t)\rangle$, which is a squared tangent vector. Moreover, it does not have the squared second derivative $\langle \mathbf{s}''(\xi,t)\mathbf{s}''(\xi,t)\rangle$, which is a squared curvature vector. In classical form, it also does not describe possible anisotropy and polarization of the loops. To overcome these difficulties, the so-called generalized Wiener distribution had been offered in the paper.[22] The generalized Wiener distribution takes into account the possible anisotropy and finite curvature. Namely, the probability $\mathcal{P}(\{\mathbf{s}(\xi,t)\})$ to find some particular configuration $\{\mathbf{s}(\xi,t)\}$ is expressed by the probability distribution functional (for details see Ref. 22)

$$\mathcal{P}(\{\mathbf{s}(\xi,t)\}) = \mathcal{N}\exp\left(-\int_0^l\int_0^l \mathbf{s}'^{\alpha}(\xi_1,t)\Lambda^{\alpha\beta}(\xi_1-\xi_2)\mathbf{s}'^{\beta}(\xi_2,t)d\xi_1 d\xi_2\right). \tag{1}$$

Here $\mathcal{N}$ is the normalizing factor and l is the length of curve. Parameters of this generalized Wiener distribution [elements of matrix $\Lambda^{\alpha\beta}(\xi_1-\xi_2)$] were taken so that some quantities (e.g., mean curvature, coefficients of anisotropy, etc.) evaluated on the basis of Eq. (1) give the values known from both experimental studies and numerical simulations. The typical form of function $\Lambda_{\alpha\beta}(\xi,\xi')$ is a smoothed δ function in a Mexican hat shape with the width equal to ξ_0. According to this model, the "average" vortex loop has a typical structure

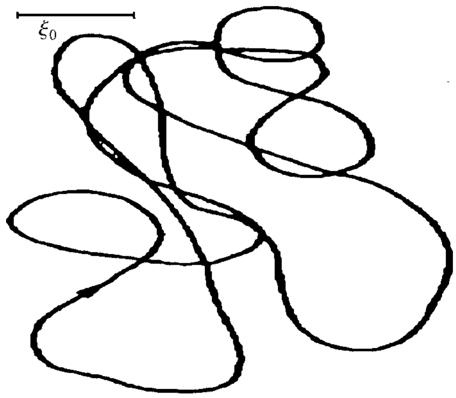

FIG. 2. Snapshot of the "average" vortex loop obtained from analysis of the statistical properties. Close ($\Delta\xi \ll \xi_0$) parts of the line are separated in 3D space by distance $\Delta\xi$. The distant parts ($\xi_0 \ll \Delta\xi$) are separated in 3D space by the distance $\sqrt{\xi_0 \Delta\xi}$, i.e., the vortex loop has the typical random-walk structure. The scale ξ_0 is depicted here in the left upper corner.

shown in Fig. 2. The average loop can be imagined as consisting of many arches with the mean radius of curvature equal ξ_0 randomly (but smoothly) connected to each other. The close parts of the loop separated (along the line) by distance $\xi_2 - \xi_1$ smaller then the mean radius of curvature ξ_0 are strongly correlated, $\langle \mathbf{s}'(\xi_1,t)\mathbf{s}'(\xi_2,t)\rangle \to 1$, ($\mathbf{s}'$ is the tangent vector) and the line is smooth. Remote parts of the line ($\xi_2 - \xi_1 \gg \xi_0$) are not correlated at all, $\langle \mathbf{s}'(\xi_1,t)\mathbf{s}'(\xi_2,t)\rangle \to 0$. Thus for large separations the vortex loop has a typical "random-walk" structure. This "semifractal" behavior satisfies the generalized Wiener distribution. Being the Gaussian (hence the name "Gaussian model"), the Wiener distribution allows for readily calculating any average functional $A(\{\mathbf{s}(\xi,t)\})$ depending on configuration $\{\mathbf{s}(\xi,t)\}$. This can be done by evaluating the following path integral:

$$\langle \mathcal{A}(\{\mathbf{s}(\xi,t)\})\rangle = \int Ds A(\{\mathbf{s}(\xi,t)\})\mathcal{P}(\{\mathbf{s}(\xi,t)\}).$$

Thus we consider the vortex tangle as a collection of vortex loops of various lengths l, which is the only degree of freedom and having the common structure constant ξ_0. Quantity ξ_0 is an important parameter of the approach. It plays the role of the "elementary step" in the theory of polymer. It is also the low cutoff of the developed approach, the theory does not describe scales smaller then ξ_0. Numerically ξ_0 is on the order of interline space $\mathcal{L}^{-1/2}$. More precisely,

$$\xi_0 = \mathcal{L}^{-1/2}/c_2(T)\sqrt{2}, \tag{2}$$

where $c_2(T)$ is the temperature-dependent function introduced by Schwarz[26] in the low-temperature limit $c_2(T) \approx 3$. The density of the loops $n(l)$ in the "space" of their lengths is defined as the number of loops (per unit volume) with lengths lying between l and $l+dl$. The distribution function $n(l,t)$ obeys the Boltzmann type "kinetic" equation.[20] From exact solution to this kinetic equation it was found,

$$n(l) = \frac{1/4c_2^2(T)}{\xi_0^{3/2}} l^{-5/2} = \frac{C_{VLD}}{\xi_0^{3/2}} l^{-5/2}, \tag{3}$$

where C_{VLD} is a constant on the order of unity. Researching of an exact solution to this kinetic equation allowed us to develop a theory of superfluid turbulence, which quantitatively describes the main features of this phenomenon.[21]

This approach turns out to be useful for the study of the inhomogeneous vortex tangle. In this case we have to impose the coordinate dependence on the distribution function and on parameter ξ_0, that is, to put $n(l,\mathbf{r},t)$ and $\xi_0(\mathbf{r},t)$, and to modify the kinetic equation with regard to the inhomogeneous situation. In fact, in this research we restrict ourselves to a more modest problem, the question of the spatial and temporal evolution of the vortex line density $\mathcal{L}(\mathbf{r},t)$. The corresponding theory can be developed in the spirit of the classical kinetic theory with the difference being that the transport processes are executed with the extended objects—vortex loops. Accordingly, the key questions is to evaluate the drift velocity V_l and the free path $\lambda(l)$ for the loop of size l.

B. Drift velocity and the free path

The drift velocity V_l is defined via an averaged quadratic velocity of the line elements (simple average velocity vanishes due to symmetry),

$$V_l = \sqrt{\left\langle \frac{1}{l}\int \dot{\mathbf{s}}(\xi)d\xi \right\rangle^2}. \tag{4}$$

The averaging $\langle\ \rangle$ should be performed with the use of the Gaussian model briefly described above. Using the local approximation approach for velocity of the line elements (see, e.g., Ref. 26), we rewrite V_l^2 as

$$V_l^2 = \left\langle \frac{\beta^2}{l^2}\int \mathbf{s}'(\xi_1)\times\mathbf{s}''(\xi_1)d\xi_1 \int \mathbf{s}'(\xi_2)\times\mathbf{s}''(\xi_2)d\xi_2 \right\rangle.$$

Quantity β is $(\kappa/4\pi)\ln(\mathcal{L}^{-1/2}/a_0)$, where a_0 is the core radius. Furthermore, using the property of the vector production, the Wick theorem for Gaussian variables and the fact that the correlator between tangent vector $\mathbf{s}'(\xi_1)$ and vector of curvature $\mathbf{s}''(\xi_1)$ vanishes, we get

$$V_l^2 = \frac{\beta^2}{l^2}\int\int \langle \mathbf{s}'(\xi_1)\mathbf{s}'(\xi_2)\rangle\langle \mathbf{s}''(\xi_1)\mathbf{s}''(\xi_2)\rangle d\xi_1 d\xi_2.$$

Exploiting further the values of correlation functions $\langle \mathbf{s}'(\xi_1)\mathbf{s}'(\xi_2)\rangle = 2\sqrt{\pi}\xi_0\delta(\xi_1-\xi_2)$ and $\langle \mathbf{s}''(\xi_1)\mathbf{s}''(\xi_2)\rangle = 1/(2\xi_0^2)$ calculated in Ref. 22, we finally find that

$$V_l = C_v\beta/\sqrt{l\xi_0}, \tag{5}$$

where C_v is a constant close to unity.

Thus, we derive the value of quadratic drift velocity of the loop of lengths l. As the length tends to value ξ_0, then velocity tends to β/ξ_0. That is justified since the loops of small size ξ_0 have a form close to rings of radius ξ_0, which have to move with the velocity β/ξ_0. On the contrary very large loops ($l \gg \xi$), consisting of many arches loops should drift with very small velocity.

The free drift continues until the loops collide with other loops with subsequent formation of larger loops. The number of collisions $P_{Col}(dt)$, per small interval dt, can be estimated from the "kinetic equation" for the distribution function $n(l)$ of density of loops in space of their lengths l (see Refs. 20 and 21). The rate of change in density $n(l)$ due to collisions is

$$\frac{\partial n(l,t)}{\partial t} = -2\int\int A(l_1,l,l_2)\delta(l_2-l_1-l)n(l)n(l_1)dl_1dl_2. \tag{6}$$

We omit the processes of the loops breakdown, due to the self-intersection. The reason is that the migration of loops is performed mainly by small loops, the large ones undergo collisions without any essential drift [moreover, the drift velocity V_l is small for large loops, see relation (5)]. The left-hand side of Eq. (6) has a meaning for number of collisions for loops of length l (number of events per unit volume and unit time). This is the quantity in which we are interested. The scattering cross section $A(l_1,l,l_2)$ describes the rate of collision of two loops with lengths l and l_2 forming the loop of length $l_1+l=l_2$. It is evaluated in the paper,[27] $A(l_1,l,l_2)=b_mV_ll_1l$. Here, b_m is the numerical factor (arising from various orientations of the line elements), approximately equal to $b_m\approx 0.2$ and $V_l=C_v\beta/\sqrt{l\xi_0}$ is the velocity of approaching vortex loops. Then, the probability $P_{Col}(dt)$ for the loop to collide with other loops (and reconnect) in small interval dt is

$$P_{Col}(dt) = \Lambda(l)dt, \tag{7}$$

where quantity Λ is evaluated with the use of relation (6)

$$\Lambda(l) = 2\int\int A(l_1,l,l_2)\delta(l_2-l_1-l)n(l_1)dl_1dl_2. \tag{8}$$

We calculate the collision probability Λ with the use of the distribution function $n(l)$ expressed by Eq. (3). Simple calculation leads to the following:

$$\Lambda(l) = 2\beta C_v b_m \mathcal{L}\sqrt{\frac{l}{\xi_0}}. \tag{9}$$

In the usual way, we conclude that probability $P(t)$ to fly the time t without collisions is

$$P(t) = \mathcal{N}\exp[-\Lambda(l)t] = \mathcal{N}\exp\left(-2\beta C_v b_m \mathcal{L} t\sqrt{\frac{l}{\xi_0}}\right), \tag{10}$$

where $\mathcal{N}$ is the normalizing factor. Using that velocity of loop $V_l=C_v\beta/\sqrt{l\xi_0}$ [see relation (5)], we get that probability $P(x)$ for the loop of length l, to fly distance x without collision is

$$P(x) = 2l\mathcal{L}\exp(-2lb_m\mathcal{L}x). \tag{11}$$

Per relation (11), the free path $\lambda(l)$ for loop of length l is

$$\lambda(l) = 1/2lb_m\mathcal{L}. \tag{12}$$

It is seen that free path $\lambda(l)$ is very small, it implies only very small loops give a significant contribution to transport processes.

C. Flux of length and the diffusion equation

Knowing the averaged velocity V_l of loops and the probability $P(x)$ (both quantities are l dependent), we can evaluate the spatial flux of the vortex line density $\mathcal{L}$, executed by the loops. The procedure is very close to the one in the classical kinetic theory, with the difference being that the carriers have different sizes, requiring additional integration over the loop lengths. Let us consider the small area element placed at some point x and oriented perpendicularly to axis x (see Fig. 2). The x component flux of the line length executed by loops of sizes l, placed in θ,φ direction (from the left and right sides, correspondingly), and remote from the area element at distance R can be written as

$$d\mathbf{j}_{\mp}(\theta,\varphi,l,R) = \ln(l,R,\theta,\varphi)(V_l\cos\theta)P(R). \tag{13}$$

In Eq. (13) $(V_l\cos\theta)$ is just the x component of the drift velocity, the factor $P(R)$ [Eq. (11)] is introduced to control an attenuation of flux due to collisions. We took the density of loops $n(l)$ and, accordingly, vortex line density $\mathcal{L}$, both of which are functions of space variable [in spatial spherical coordinates $n=n(l,R,\theta,\varphi)$, $\mathcal{L}=\mathcal{L}(l,R,\theta,\varphi)$]. In the spirit of classical kinetic theory, we assume the local equilibrium is established. In particular, the parameter ξ_0 [also spatially dependent $\xi_0=\xi_0(R,\theta,\varphi)$] is not independent but takes the "equilibrium" value $\xi_0=\mathcal{L}^{-1/2}/c_2(T)\sqrt{2}$, given by relation (2), (see, however, Ref. [28]). We also take the local density of loops, with the equilibrium value $n(l)$ expressed by Eq. (3). The net flux through the area element is

$$\mathbf{J}_x = \int[d\mathbf{j}_+(\theta,\varphi,l,R) - d\mathbf{j}_-(\theta,\varphi,l,R)]\frac{\sin\theta d\theta d\varphi}{4\pi}dldR. \tag{14}$$

Assuming that nonuniformity is along the axis x and substituting expressions for the drift velocity V_l [Eq. (5)] and for the probability of collisionless flight $P(R)$ [Eq. (11)], we rewrite [Eq. (14)] in the following form:

$$\mathbf{J}_x = \frac{1}{4\pi}\int l\frac{\partial n(l,x,t)}{\partial x}(2R\cos\theta)\left(\frac{\beta}{\sqrt{l\xi_0}}\cos\theta\right) \times[2lb_m\mathcal{L}\exp(-2lb_m\mathcal{L}R)]\sin\theta d\theta d\varphi dldR. \tag{15}$$

Using relations (2) and (3), the fact that integral $\int \ln(l)dl=\mathcal{L}$, we can get rid of quantities $\xi_0(x,t)$ and $n(l,x,t)$, and express all the terms via vortex line density $\mathcal{L}(x,t)$,

$$\mathbf{J}_x = -D_v\,\partial\mathcal{L}/\partial x. \tag{16}$$

Correspondingly, in the general [three-dimensional (3D)] case, the spatial-temporal evolution of quantity $\mathcal{L}$ obeys the diffusion type equation

$$\frac{\partial \mathcal{L}}{\partial t} = D_v \nabla^2 \mathcal{L}, \tag{17}$$

where the diffusion coefficient D_v is equal to $C_d\kappa$. Our approach is rather crude to claim a good quantitative description. However, if we are to adopt the data grounded in the exact solution to the Boltzmann-type kinetic equation (Ref. 21) we conclude that $C_d \approx 2.2$. As discussed above, the result that evolution of the vortex line density obeys the diffusion type equation, with the diffusion constant on the order of κ, was expected. A point of particular interest is related to the prefactor C_d. The only former quantitative result was obtained numerically in the paper.[5] The authors suggested that $C_d \approx 0.1$, which is about 20 times smaller than our value. We will discuss the probable reason for this large discrepancy later. The result obtained is valid for zero temperature. The case of finite temperature requires separate consideration since all quantities from the drift velocity V_l up to coefficient $c_2(T)$ between curvature and interline space change due to the presence of the normal component. We think, however, that the change in the diffusion coefficient D_v should not be too excessive.

Taking into account that the value $D_v \approx 2.2\kappa$ is the key outcome of this work, it would be instructive to derive the diffusion constant D_v from transparent qualitative consideration without appealing to the Gaussian model. From the classical kinetic theory (for the flux of ordinary molecules) it is common knowledge that the phenomenological value of the diffusion coefficient is $(1/3)V_l\lambda_{free}$ (λ_{free} is the free path) and this value practically coincides with the one obtained from the rigorous theory (Boltzmann equation). Thus, having the values of the average velocity V_l and the free path l_{free}, one can easily estimate the diffusion constant.

Let us start with the free path. The probability $P(l_1, l, \Delta t)$ (per unit volume) that a rod of length l intersects another rod l_1, in time interval Δt, should be proportional to the production of their sizes and approaching velocity V_l, $P(l_1, l, \Delta t) = b_m V_l l_1 l \Delta t$. Here, $b_m \approx 0.2$ is a geometric factor, taking into account the different orientation of loops (in the context of colliding cosmic strings, this result is obtained in Ref. 25). Further, we have to collect contributions from all encountered loops with length l_1 with the use of the distribution function $n(l) = Cl^{-5/2}$. Having relation $P(l, \Delta t) = \Lambda(l)\Delta t$, for the probability that the rod (of the length l) collides in the time interval Δt, one can conclude that the probability $P(x)$, for the rod of length l to fly distance x without collision, is $P(x) = 2l\mathcal{L}\exp(-2lb_m\mathcal{L}x)$. So, the free path for loop of length l is $1/2lb_m\mathcal{L}$.

The drift velocity V_l can be derived from the following qualitative reasoning. Successively considering that the average loop consists of $n = l/\xi_0$ arches, with the mean radius of curvature equal to ξ_0 randomly (but smoothly) connected to each other, we can take its velocity as the resulting velocity of all arches composing the loop. Since the arches are randomly connected to each other and have the velocity as for rings, $V_{arch} = \beta/\xi_0$ (directed along the normal), the resulting averaged velocity is the "random walking" average, $V_l = \frac{1}{n}\sqrt{n}V_{arch} = \beta/\sqrt{l\xi_0}$.

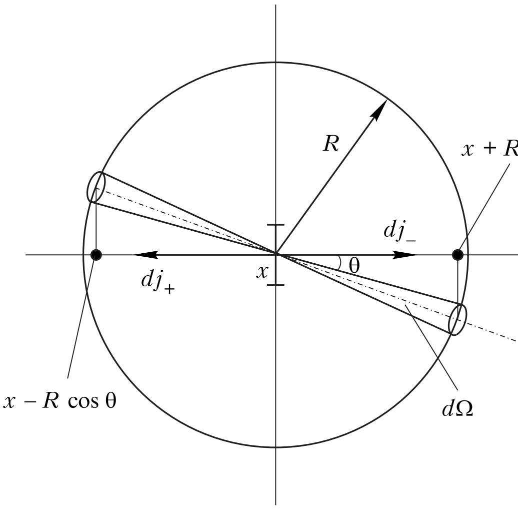

FIG. 3. The net flux of length through the small area element placed in $x=0$ and orientated perpendicularly to axis x.

Combining the obtained results on the free path and average quadratic velocity, we can conclude that the diffusion coefficient of loops with length l is $D(l) = \beta/(6\sqrt{l\xi_0}\mathcal{L}lb_m)$. Collecting contributions from loops of different sizes, we obtained that the diffusion coefficient D is about $\kappa c_2^2/12b_m \approx 3.75\kappa$, which is approximately 1.5 times larger than value 2.2κ, obtained with the use of the Gaussian model.

III. BOUNDARY CONDITIONS

The boundary conditions depend on the specific statement of the problem. We will consider three different situations.

(1) *Smearing of the tangle.* First, let us consider the case where the vortex tangle is placed in some restricted domain of superfluid helium. Let us also consider that vortex line density $\mathcal{L}$ is not too high, allowing the relatively large loop to be radiated. Moving faster, the smaller loops run down the larger loops, colliding and reconnecting with them. In this manner, outside of initial domain, the well-developed tangle is formed. This, secondary, vortex tangle smoothly joins the initial tangle inside the domain. This implies that, in this case, no boundary conditions are required at all and evolution of the vortex line density obeys Eq. (17) in infinite space, with initial distribution $\mathcal{L}(\mathbf{r}, 0)$, inside this domain.

(2) *Radiation of loops.* The second situation is accomplished when the radiated vortex loops do not form a sufficiently developed enough vortex tangle to organize the back flux (into volume). It can happen, for instance, if the initial tangle is very dense, causing it to radiate only very small loops, which rapidly propagate. These loops run away without interaction with each other, and with the initial tangle, from where they are radiated. Another hypothetical variant is if there is some trap on the boundary, absorbing vortex loops. The boundary conditions can be found, assuming that diffusivelike flux of length near boundary $\mathbf{J} = -D_v \nabla \mathcal{L}(x_b, t)$ (x_b is the coordinates of the boundary) coincides with the flux executed by vortex loops, radiated through the (right) boundary

$\mathbf{J}_r(x_b,t)$. The latter is evaluated by an update of Eqs. (14) and (15), where we have to take "noncompensating" flux, only from the (left) half space (visualize the left half of Fig. 3),

$$\begin{aligned}\mathbf{J}_{rad} &= \frac{1}{4\pi}\int l[n(l,x_b+R\cos\theta)] \\ &\quad\times[\mathbf{v}(l)\cos\theta]P(R)\sin\theta d\theta d\varphi dl dR \\ &= \frac{1}{4\pi}\int l\left[n(l,x_b,t)+\frac{\partial n(l,x,t)}{\partial x}(R\cos\theta)\right] \\ &\quad\times\left\{\left(\frac{\beta}{\sqrt{l\xi_0}}\cos\theta\right)[2lb_m\mathcal{L}\exp(-2lb_m\mathcal{L}R)]\right. \\ &\quad\left.\times\sin\theta d\theta d\varphi dl dR.\right\} \end{aligned} \tag{18}$$

The contribution from the second term in square brackets can be evaluated as early [see relations (15) and (16)] giving the flux,

$$-\frac{1}{2}D_v\nabla\mathcal{L}(x_b,t). \tag{19}$$

To evaluate the contribution from the first term in square brackets, we have to use again Eqs. (15) and (16). Simple calculation leads to result

$$C_{rad}\beta\mathcal{L}^{3/2}, \tag{20}$$

with $C_{rad}\approx 0.47$. The sum of Eqs. (19) and (20) gives the flux, formed with the radiated and escaped loops. It should be equal to the flow $-D_v\nabla\mathcal{L}(x_b,t)$, which provides the loops in the region near the border. Then we finally have the following boundary condition:

$$C_{rad}\mathcal{L}^{3/2}+\frac{1}{2}D_v\nabla\mathcal{L}(x_b,t)=0. \tag{21}$$

For the left boundary there should be a minus sign.

(3) *Solid walls.* In the case of a solid wall, which corresponds to some experiments, the situation is more involved. Vortices can annihilate on the solid wall, they can undergo pinning and depinning, with the back radiation of vortices entering the bulk of helium. Surely, this requires a special treatment, which goes beyond the scope of the work. One possibility is to consider the solid wall as a "partial" trap, which catches the loops and re-emits a part of them back into the bulk. Formally, it can be written as condition (21) with additional term $\mathbf{J}_{back}$ describing the back flux. Without detailed analysis, it can be assumed that the back flux is proportional to the vortex line density $\mathcal{L}(x_b,t)$ on boundary, $\mathbf{J}_{back}=-C_{back}\mathcal{L}(x_b,t)$ with coefficient C_{back} depending on the dynamics of the lines on the wall (jumps between pinning sites, Kelvin waves dynamics near the wall, etc.). Thus, the boundary condition (21) for pure trap can be written in form

$$C_{rad}\mathcal{L}^{3/2}+\frac{1}{2}D_v\nabla\mathcal{L}(x_b,t)-C_{back}\mathcal{L}=0. \tag{22}$$

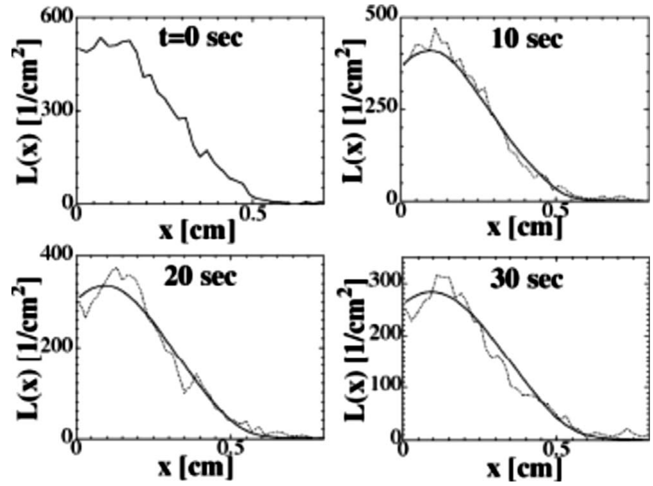

FIG. 4. Development of the vortex line density distribution. The dotted line shows the results of the numerical simulation while the solid lines obtained from solution of the diffusion equation with the diffusion constant is equal to $C_d\approx 0.1\times 10^{-3}$ cm^2/s and with the auxiliary term $-\chi_2(\kappa/2\pi)\mathcal{L}^2$ [Tsubota *et al.* (Ref. 5)].

IV. DECAY OF THE VORTEX TANGLE

Numerical simulations

Let us discuss how the developed approach can be applied to the problem of the vortex tangle decay at zero temperature. It is clear that escape of the loops from the bulk results in the attenuation of the vortex line density. As discussed above, the contribution of diffusion (or radiation of loops) is usually ignored, mainly due to the small value of the diffusion constant offered in the paper.[5] The authors offered the value $\approx 0.1\kappa$, which is approximately 20 times smaller than the $D_v=2.2\kappa$, obtained in the present paper. Let us discuss the probable reason for this large discrepancy. In the paper[5] it was studied one-dimensional evolution (spatial spreading) of the vortex tangle, concentrated initially in a domain of space and having nonuniform distribution as it is depicted in the first picture in Fig. 4. To describe this evolution of the vortex line density it was suggested that the quantity $\mathcal{L}(x,t)$ obeys Eq. (17), with the additional term $-\chi_2(\kappa/2\pi)\mathcal{L}^2$, describing the "homogeneous decay" of the vortex tangle.[29] In turn, this term was introduced to describe the decay of the vortex tangle in previous numerical simulation made by Tsubota *et al.*[15] It was done merely to adjust the Vinen equation without profound analysis of the mechanisms of the decay.

Let us briefly analyze the situation presented in the paper[15] and demonstrate that none of the currently discussed mechanisms of the "homogeneous" decay of the vortex tangle at zero temperature can be applied to this work.

Kelvin waves cascade with the subsequent dissipation. Kelvin waves cascade could not be a reason for the vortex tangle decay in numerical simulation,[15] simply because the space resolution $\Delta\xi=2\times 10^{-2}$ cm was too large to monitor the region of large wave numbers, required for observation of a generation of higher harmonics.

Acoustic radiation. Similarly, the acoustic radiation could not be a reason for the vortex tangle decay because the compressibility had not been included in the numerical scheme.

Loss of the line length during reconnection. Real effect of the length loss can be obtained only on the basis of more rigorous theory, e.g., with the use of the Gross-Pitaevskii equation. It is known, however, that an artificial loss of length is possible due to realization of the reconnection procedure. This effect, however, should depend on the space resolution, whereas it was proven that the rate of decay did not depend on $\Delta\xi$ (see Fig. 11 of the paper[15]).

Feynman cascade of consequent breaking down of vortex loops. The breaking down of vortex loops is a very important ingredient in the whole vortex tangle evolution. This process was imitated in Ref. 15 with elimination of very small loops (with sizes of several $\Delta\xi$). Unfortunately, exact monitoring of these eliminations had not been performed. However, it is clear that very small loops can appear, only in the processes of self-intersection of larger loops. The total number of reconnections of this type (self-reconnection) was estimated in Ref. 15 (see Table I of this paper). Even this amount is too small to describe the decay of vortex line density observed in the numerical experiment.

To resume, one concludes that none of the discussed mechanisms could be the reason of the homogeneous decay of the vortex tangle in numerical simulation.[15] Thus, the nature of attenuation of the vortex line density in Ref. 15 [and, correspondingly, the nature of the term $-\chi_2(\kappa/2\pi)\mathcal{L}^2$] remained unclear. The question of diffusivelike decay was not discussed in this paper, although the authors did describe how vortex loops escaped from the volume and stuck to the walls. In light of the foregoing, we assume that the vortex tangle decay observed in Ref. 15 occurred due to the escape of the vortex loops and the latter was implemented with a diffusion mechanism. The same assumption concerns the work,[5] where a similar numerical procedure had been used. Thus, we assert that the decay of the vortex tangle in both cited works can be described essentially as a process of diffusion, without the additional term $-\chi_2(\kappa/2\pi)\mathcal{L}^2$. *At the same time, because of this additional term, the contribution of diffusion to the whole decay would be significantly underestimated in work.*[5]

To confirm our assumptions, we calculated the spatial-temporal evolution of the vortex line density $\mathcal{L}(\mathbf{r},t)$ [under the conditions of numerical experiment (Refs. 5 and 15)], using Eq. (17), with the diffusion coefficient equal to 2.2κ. In the paper[5] it was studied evolution of the vortex tangle confined to a 1 cm cube. When a random tangle was developed, the counterflow is turned off and the temperature reduced to zero, vortices with parts in the right-hand half of the cube are removed, and the evolution of the remaining vortices is followed, now with a fully nonlocal Biot-Savart dynamics. To model these conditions we studied one-dimensional evolution (spatial spreading) of the vortex tangle, concentrated initially in a domain of space and having nonuniform distribution as it is depicted in the first picture in Fig. 4. We accepted the boundary conditions of the first kind (see above) since there was free smearing of the tangle into the vortex free volume filled with helium II. In this situation the classical solution (see, e.g., Ref. 30)

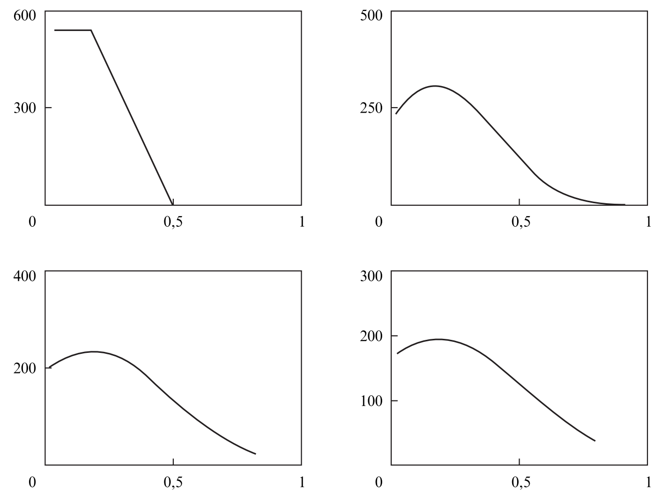

FIG. 5. Evolution of the vortex line density calculated with the use of Eq. (17) without the auxiliary term, the diffusion constant was equal to $C_d \approx 2.2\times 10^{-3}\ \mathrm{cm}^2/\mathrm{s}$.

$$\mathcal{L}(x,t) = \frac{1}{2\sqrt{\pi D_v t}}\int_{-\infty}^{\infty} \mathcal{L}(x,0)\exp\left[-\frac{(x-\eta)^2}{4D_v t}\right]d\eta \quad (23)$$

can be applied. The initial distribution $\mathcal{L}(x,0)$ corresponds to the first (upper left) picture in Fig. 4. The result of calculation basing on relation (23) is depicted in picture of Fig. 5.

The situation in the paper[15] is somewhat different. Initially, the vortex tangle uniformly occupied a cubic volume with an edge equal 1 cm. It was created by counterflowing helium II at finite (about 1.6 K) temperature. Periodic boundary conditions were applied at the faces normal to the flow; the other faces were taken as solid. Then the mutual friction coefficient was set to be equal to zero and the counterflow was switched off. After that the decay of superfluid turbulence was observed. Attenuation of vortex line density is depicted in the upper picture of Fig. 6. Periodic boundary condition in one direction mimicked an infinite uniform distribution along this axis. Thus, the problem turned out to be essentially two dimensional. We calculated the two-dimensional evolution of the vortex line density in the 1 cm square resolving numerically Eq. (17) with the boundary conditions of third kind (solid walls). As discussed above, we can (in the frame of the present paper) determine the boundary condition only up to the coefficient of re-emission C_{back}. Considering it as a fitting parameter, we have chosen the value of $C_{back}\approx 4$. The problem had been resolved numerically and the result is depicted in the lower picture of Fig. 6. It should be noted that the solution is not too sensitive to choice of the fitting parameter. For example, the changed in the fitting parameter $C_{back}\approx 4$ from 0.5 to 10 led to a change in the final value of the vortex line density (at $t=100$ s) from 90 to 150 cm^{-2}. Comparison of Figs. 4 and 5 as well as the upper and lower pictures of Fig. 6, enabled us to conclude that the diffusion spreading (with diffusion constant $D_v \approx 2.2\times 10^{-3}\ \mathrm{cm}^2/\mathrm{s}$) describes satisfactorily the evolution of the vortex tangle, without any additional term. Moreover, as discussed above, there are no reasons for the homogeneous decay in the numerical studies.[5,15]

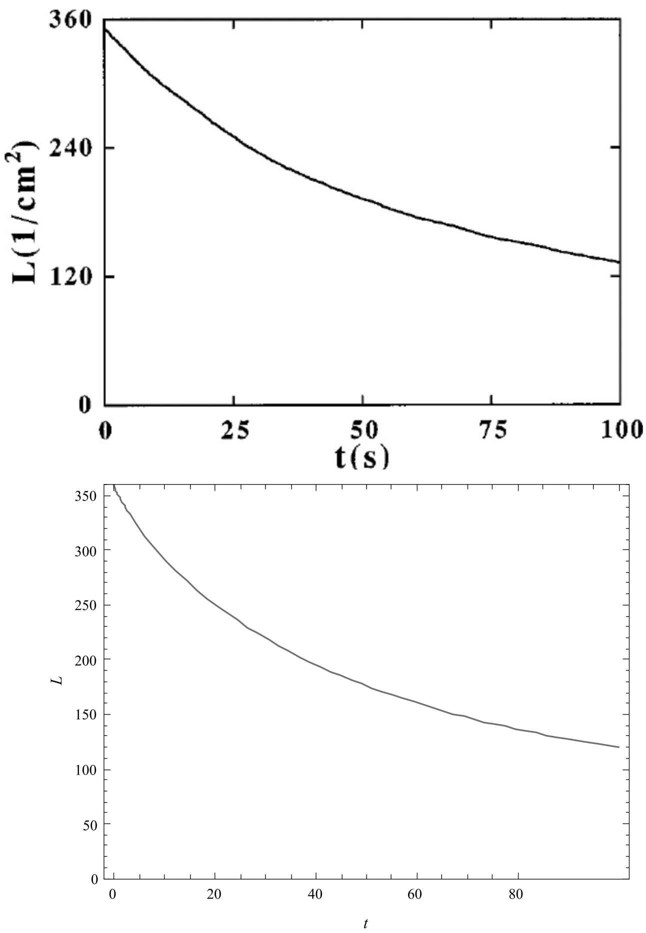

FIG. 6. In the upper picture it is depicted the attenuation of vortex line density obtained in numerical experiment (Ref. 15). In the lower picture it is depicted the same quantity calculated with the use of Eq. (17) without the auxiliary term, the diffusion constant was equal to $C_d \approx 2.2 \times 10^{-3}$ cm^2/s.

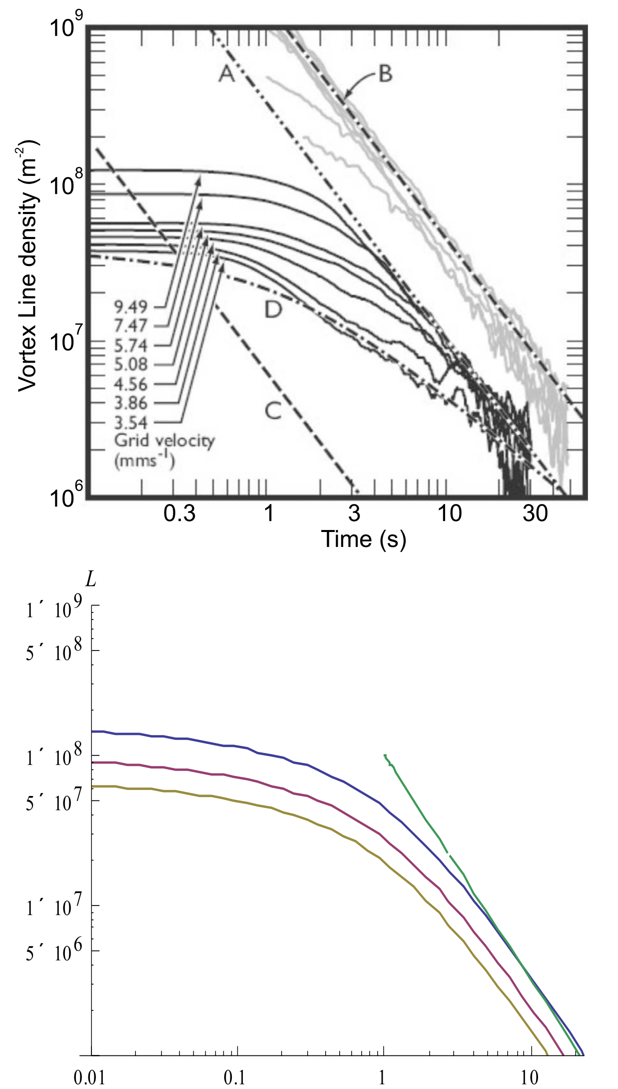

FIG. 7. (Color online) Comparison with experiment (Ref. 17). See the text.

V. EXPERIMENTAL RESULTS

Let us now discuss two recent experiments on decay of the vortex tangle at very low temperatures.[17,18] The authors reported attenuation of vortex line density in superfluid turbulent helium (^{3}He-B in the paper[17] and ^{4}He in the paper[18]). They attribute the decay of the vortex tangle to the classical turbulence mechanisms. Without discussing this variant, we would like to simply estimate the contribution into attenuation of the vortex line density due to the pure diffusion mechanism.

In the upper picture of Fig. 7, we displayed Fig. 2 of work,[17] showing results of measurements on the temporal behavior of the average vortex line density $\mathcal{L}(t)$ (solid curves, see Ref. 17 for details). We calculated the same quantity resolving the diffusion Eq. (17), with the use of the first kind (see above) boundary condition, which corresponds to the smearing of the vortex tangle into ambient space. It is known that for this condition the solution of three-dimensional problem is just production of three one-dimensional solutions of form (23). Therefore, the averaged vortex line density $\mathcal{L}(t)$ can be evaluated as follows:

$$\mathcal{L}(t) = \mathcal{L}(x,0)\left\{\frac{1}{2x_0}\int_{-x_0}^{x_0} dx\left[erf\left(\frac{x_0 - x}{\sqrt{4D_v t}}\right) + erf\left(\frac{x_0 + x}{\sqrt{4D_v t}}\right)\right]\right\}^3. \tag{24}$$

Here $\mathcal{L}(x,0)$ is the initial distribution of the vortex line density, which is supposed to be uniform in the domain $(-x_0 < x < x_0)$ and $erf(z)$ is the error function. In the lower picture of Fig. 7 we depicted the evolution of the averaged vortex line density $\mathcal{L}(t)$ for three different initial conditions. The lines were obtained evaluating Eq. (24). The straight line in the lower picture exactly corresponds to line A in the upper picture (this line was named by the authors of paper[17] as a "limiting behavior"). It is obvious that there is substantial correlation between experimental data and theoretical predictions, except for the fact that our curves do not collapse in universal limiting behavior at the late-time interval but are slightly separated.

In contrast to work,[17] in the paper,[18] the decay of vortex tangle in He II was observed in the closed cube with solid walls. In the upper picture of Fig. 8, it is depicted the temporal behavior of the average vortex line density $\mathcal{L}_{av}(t)$ is depicted. We calculated the same dependence on the basis of the diffusion Eq. (17), with the third kind boundary (solid walls). As discussed above, we can (in the frame of the present paper) determine the boundary condition only up to the coefficient of re-emission C_{back}. In example with the numerical simulation[15] we used $C_{back}=4$. In principle, it is not necessary that the constant describing the back radiation of the loops is the same in different experiments. Indeed, this quantity depends on the properties of surface and the latter can be different in various experiments. Nevertheless we used again the value $C_{back}=4$. The fully three-dimensional problem was resolved numerically, the result shown in the lower picture of Fig. 8. Considering it as a fitting parameter, we have chosen the value of $C_{back}\approx 0.9$. It can be seen that the decay of the vortex tangle, due to diffusion, reproduces some feature observed in the experiment. There is one quite interesting by-product of the consideration exposed above. For small values of vortex line density the bend appears on curve $\mathcal{L}(t)$ (see the lower picture of Fig. 8). It corresponds to the fact that the diffusivelike flux of the length vanishes [because of vanishing the gradient $\nabla\mathcal{L}(x_b,t)$], and only the first and third terms in the boundary condition (22) survive. Let us recall that these terms correspond to radiation of loops, from the bulk to the boundary and to the re-emission of loops from the wall, into the bulk of helium. Comparing these terms (for the chosen value $C_{back}\approx 0.9$), we discover that equilibrium is reached for the value of the vortex line density $\mathcal{L}$ of order 50–100 cm^{-2}. This value can be considered as a "background" value of pre-existing vortices in helium.

Let us now discuss the comparison between experimental data and our predictions. First of all, the whole qualitative behavior of lines agrees with diffusivelike attenuation. In particular, there is a plateau, which is changed with the fast decay of the tangle. Full decay of the superfluid turbulence occurs in times, which are in a very good agreement, predicted on the basis of the diffusion approach elaborated here. The slope of the curve in the interval of the most intensive decrease shows the dependence close to $\sim t^{-3/2}$, which is also typical for diffusion.

Resuming this section we can conclude that our theoretical predictions have good agreement with the experimental data. Nevertheless, we would like to stress that the study does not claim to be an exhaustive explanation for experiments [Refs. 17 and 18]. First of all, there is one feature, which we are not in a position to describe. Our approach does not give an answer why, at large moments of time, all curves (independently on initial values), collapse into single behavior. One of the reasons for this discrepancy would be that in the region where this limiting behavior is observed, the vortex line density is small and the vortex tangle is extremely diluted, and are only few lines in the whole volume exist. For the diluted vortex tangle, the model based on kinetics of merging and splitting vortex loops is not satisfactory. Instead of a set of the vortex loops of different sizes, few long lines stretching for the whole volume appear in the system. Second, we do not consider here, the case where the vortex lines are strongly polarized so that the coarse-grained motion induced by the bundles of filaments imitates classical

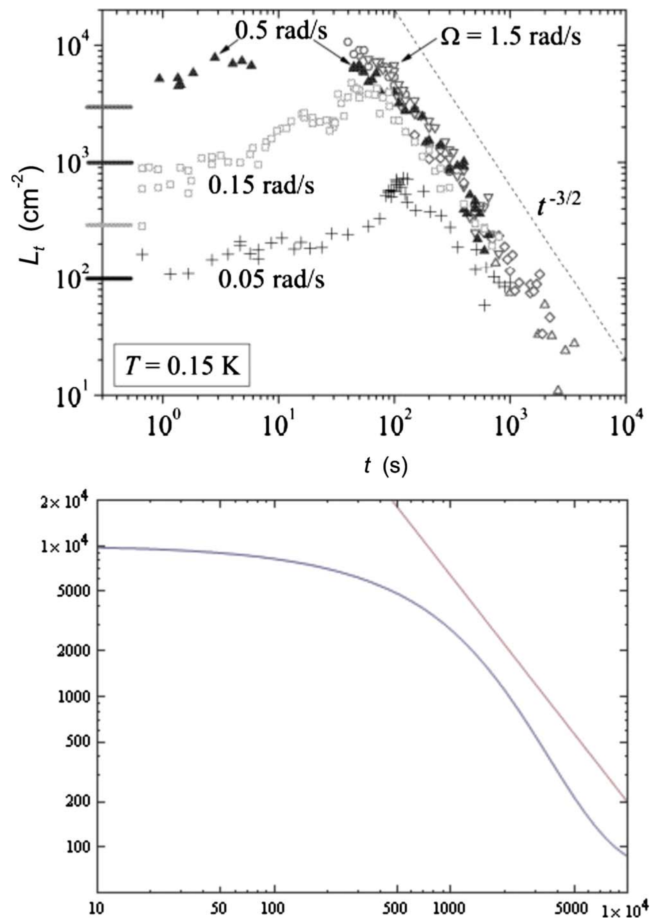

FIG. 8. (Color online) Comparison with experiment (Ref. 18). See the text.

(Kolmogorov) turbulence (see Refs. 10, 31, and 32). We consider here, the case of the so-called Vinen turbulence, where the vortex loops are highly disordered, with the zero mean vorticity. At the same time, it would not be ruled out that superfluid turbulence observed in works[17] and Ref. 18 belongs to the first case of polarized vortex lines. Therefore, the conclusion to this section can be formulated as "the decay of the superfluid turbulence is a very involved and unclear process and the possibility of diffusion affecting the decays reported in Refs. 17 and 18 ought to be taken into account." Especially it concerns the situation when decay of the superfluid turbulence serves as the basis for the idea about similarity between classical and quantum turbulence.

VI. CONCLUSION

In summary, the theory describing the evolution of the inhomogeneous vortex tangle at zero temperature was developed on the bases of kinetics of merging and splitting vortex loops. Using the Gaussian model for vortex loops we calculated the (size-dependent) free path and mean quadratic velocity of vortex loops. With the use of these quantities we calculated the flux of the vortex line density $\mathcal{L}(x,t)$ in an inhomogeneous vortex tangle and demonstrated that under certain circumstances it satisfies the diffusionlike equation with the coefficient equal approximately to 2.2κ. We used this equation to describe the decay of the vortex tangle at very low temperature. We compared the solution with the recent experiments on the decay of the superfluid turbulence. There was agreement with the experimental data allowing us to conclude that the diffusion processes give a significant contribution in the free decay of the vortex tangle at the absence of the normal component. The decay of superfluid turbulence is not the only application of the theory developed. Some other obvious applications can be related with the motion of turbulent fronts and "plugs" or, e.g., with direct observation of the spread of turbulence in the bulk. These applications can be experimentally tested and will be investigated in the future.

ACKNOWLEDGMENTS

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[1] H. van Beelen, W. van Joolingen, and K. Yamada, Physica B **153**, 248 (1988).
[2] J. A. Geurst, Physica B **154**, 327 (1989).
[3] S. K. Nemirovskii and W. Fiszdon, Rev. Mod. Phys. **67**, 37 (1995).
[4] M. S. Mongiovì and D. Jou, Phys. Rev. B **75**, 024507 (2007).
[5] M. Tsubota, T. Araki, and W. F. Vinen, Physica B **329-333**, 224 (2003).
[6] C. F. Barenghi and D. C. Samuels, Phys. Rev. Lett. **89**, 155302 (2002).
[7] D. I. Bradley, D. O. Clubb, S. N. Fisher, A. M. Guenault, R. P. Haley, C. J. Matthews, G. R. Pickett, V. Tsepelin, and K. Zaki, Phys. Rev. Lett. **95**, 035302 (2005).
[8] C. Nore, M. Abid, and M. E. Brachet, Phys. Rev. Lett. **78**, 3896 (1997).
[9] S. R. Stalp, L. Skrbek, and R. J. Donnelly, Phys. Rev. Lett. **82**, 4831 (1999).
[10] W. F. Vinen, Phys. Rev. B **61**, 1410 (2000).
[11] T. Araki, M. Tsubota, and S. K. Nemirovskii, Phys. Rev. Lett. **89**, 145301 (2002).
[12] T. V. Chagovets, A. V. Gordeev, and L. Skrbek, Phys. Rev. E **76**, 027301 (2007).
[13] P. M. Walmsley and A. I. Golov, Phys. Rev. Lett. **100**, 245301 (2008).
[14] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, 1996).
[15] M. Tsubota, T. Araki, and S. K. Nemirovskii, Phys. Rev. B **62**, 11751 (2000).
[16] S. I. Davis, P. C. Hendry, and P. V. E. McClintock, Physica B **280**, 43 (2000).
[17] D. I. Bradley, D. O. Clubb, S. N. Fisher, A. M. Guenault, R. P. Haley, C. J. Matthews, G. R. Pickett, V. Tsepelin, and K. Zaki, Phys. Rev. Lett. **96**, 035301 (2006).
[18] P. M. Walmsley, A. I. Golov, H. E. Hall, A. A. Levchenko, and W. F. Vinen, Phys. Rev. Lett. **99**, 265302 (2007).
[19] C. F. Barenghi, Physica D **237**, 2195 (2008).
[20] S. K. Nemirovskii, Phys. Rev. Lett. **96**, 015301 (2006).
[21] S. K. Nemirovskii, Phys. Rev. B **77**, 214509 (2008).
[22] S. K. Nemirovskii, Phys. Rev. B **57**, 5972 (1998).
[23] H. Kleinert, *Gauge Fields in Condenced Matter Physics* (World Scientific, Singapore, 1990).
[24] F. W. Wiegel, *Introduction to Path-Intregral Methods in Physics and Polymer Sciencs* (World Scientific, Singapore, 1986).
[25] E. J. Copeland, T. W. B. Kibble, and D. A. Steer, Phys. Rev. D **58**, 043508 (1998).
[26] K. W. Schwarz, Phys. Rev. B **38**, 2398 (1988).
[27] S. K. Nemirovskii, J. Low Temp. Phys., **142**, 769 (2006).
[28] In principal there is a possible artificial situation when parameter ξ_0 is independent. This case can be realized while injecting the loops (rings) of a fixed size into volume. One more case can appear when the loops are emitted as a result of overlapping Kelvin wave sand the mean radius of curvature ξ_0 should be connected to the spectrum of Kelvin waves (Ref. 33). In this case we have to consider the quantaty ξ_0 to be an additional variable and the whole approach should be significantly modernized.
[29] We use the term homogeneous decay, to describe the attenuation of the vortex tangle at zero temperature due to mechanisms such as a cascadelike break down of the loops, a Kelvin waves cascade, acoustic radiation, reconnection loss, etc., as discussed in Ref. 19. This should be distinguished it from the diffusionlike attenuation of the vortex line density described in the present paper.
[30] H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (Clarendon Press, Oxford, 1984).
[31] G. E. Volovik, JETP Lett. **78**, 553 (2003).
[32] L. Skrbek, Pis'ma Zh. Eksp. Teor. Fiz., **80**, 541 (2004).
[33] B. V. Svistunov, Phys. Rev. B **52**, 3647 (1995).